A Fully Covariant Description of CMB Anisotropies

Peter K. S. Dunsby

Department of Applied Mathematics, University of Cape Town, Rondebosch 7700, Cape Town, South Africa and

Department of Mathematics, Statistics, and Computing Science, Dalhousie University, Halifax, Nova Scotia, Canada

Abstract. Starting from the exact non - linear description of matter and radiation, a fully covariant and gauge - invariant formula for the observed temperature anisotropy of the cosmic microwave background (CBR) radiation, expressed in terms of the electric (E_{ab}) and magnetic (H_{ab}) parts of the Weyl tensor, is obtained by integrating photon geodesics from last scattering to the point of observation today. This improves and extends earlier work by Russ *et al* where a similar formula was obtained by taking first order variations of the redshift. In the case of scalar (density) perturbations, E_{ab} is related to the harmonic components of the gravitational potential Φ_k and the usual dominant Sachs - Wolfe contribution $\delta T_R/\bar{T}_R \sim \Phi_k$ to the temperature anisotropy is recovered, together with contributions due to the time variation of the potential (Rees - Sciama effect), entropy and velocity perturbations at last scattering and a pressure suppression term important in low density universes. We also explicitly demonstrate the validity of assuming that the perturbations are adiabatic at decoupling and show that if the surface of last scattering is correctly placed and the background universe model is taken to be a flat dust dominated Friedmann - Robertson - Walker model (FRW), then the large scale temperature anisotropy can be interpreted as being due to the motion of the matter relative to the surface of constant temperature which defines the surface of last scattering on those scales.

1. Introduction

The study of the cosmic microwave background (CMB) radiation is the corner stone of modern Big Bang cosmology and has led to its widespread acceptance. Over the last few years improved measurements of the temperature spectrum and anisotropy has led to a better understanding of the origin and evolution of large scale structure in the universe.

The physical basis for using anisotropies in the CMB to constrain competing theories of galaxy formation is the Sachs - Wolfe effect [25]. The basic assumption is that the

observed CMB photons travel to us without significant interaction with matter from a redshift of about $z = 1200$. At redshifts greater than this, the universe is ionized and photons are coupled to the electron - baryon plasma through Thompson scattering.

The process of decoupling i.e. the transition of the CMB from a collisional regime to being free photons does not take place instantaneously. The thickness of the decoupling shell Δz is approximately 1/15 of the mean redshift, which from our point of view as observers is relatively narrow [16], so for many purposes we can treat this shell as a sharp surface, called the *surface of last scattering* (SLS). The correct way of placing this surface is by determining where the optical depth due to Thompson scattering is unity [22, 29, 26, 10]. This occurs, to first order, where the radiation temperature (which is equal to the matter temperature in the strongly coupled region prior to decoupling) reaches the matter ionization temperature, so if we take decoupling as happening essentially instantaneously, the SLS is, to good approximation, a surface of constant radiation temperature.

Causally connected regions at the SLS, as viewed by an observer today, subtend an angle $\theta \sim 2\sqrt{\Omega_0}$, where Ω_0 is the present value of the density parameter, so large scale anisotropies on angular scales greater than 7 degrees are unaffected by the small scale physics of decoupling, and so represent primordial perturbations. These anisotropies arise because photons traveling from the SLS are red - shifted slightly more than they would be in a perfectly homogeneous universe as a result of having to climb from an increased gravitational potential due to density perturbations over the surface.

The calculation of CMB anisotropies on angular scales larger than a degree, corresponding to scales larger than the Hubble radius at the time of decoupling, is very simple in principle, however in practice the formulation of a completely self - consistent theoretical picture has been plagued by a misunderstanding of the observational meaning of certain temperature perturbation measures and whether or not these measures are gauge - invariant. Furthermore many approximations are made in deriving the various contributions to the anisotropy, without adequate consideration of whether or not these approximations are justifiable or consistent with each other.

In this paper we attempt to address some of the above problems, by deriving from first principles a formula for the CMB anisotropy which is both simpler and easier to interpret than many of the usual treatments. The approach we take is to integrate photon geodesics from the time of last scattering to today, obtaining a general (model independent) formula for the observed temperature in a given direction in the sky and expressing the result in terms of covariantly defined gauge - invariant quantities. The temperature anisotropy is then given by subtracting this result for two independent directions corresponding to different points of emission on the SLS. This improves and extends earlier work by Russ *et al* [24] by (i) starting from a general non - linear treatment of the geodesic equation and linearizing to obtain the *almost* FRW result, rather than

taking first order variations of the redshift; (ii) correctly defining the temperature perturbation measure $\delta T/\bar{T}$; and (iii) performing a two-fluid analysis instead of treating the radiation as a test field on a single-fluid background.

Following this, we demonstrate that given a number of "standard" assumptions, the classical Sachs-Wolfe result is recovered and we also consider in detail the validity of the assumption that the perturbations in the total energy density of the photon-baryon fluid at decoupling are adiabatic.

Conventions on signature, Riemann and Ricci tensors are as in [9], and the speed of light is taken to be unity ($c = 1$). Standard General Relativity is assumed, with Einstein's equations in the form $G_{ab} = \kappa T_{ab}$ where G_{ab} is the usual Einstein tensor, $\kappa = 8\pi G$ is the gravitational constant and T_{ab} is the energy momentum tensor of the matter. Most of the notation is the same as in [2, 4] and any changes are stated in the text.

2. Basic equations and notation

For the sake of self-consistency and clarity, we will first summarize the covariant approach to cosmology.

2.1. The covariant approach

As in Hawking [14] and Ellis [8, 9], the hydrodynamic fluid 4-velocity (tangent to the worldlines of fundamental observers in the universe) is $u^a = dx^a/dt$ ($u^a u_a = -1$), where t is the proper time along the flow lines. The projection tensor into the tangent three-spaces orthogonal to u^a (the local rest frame of these observers) is

$$h_{ab} = g_{ab} + u_a u_b \; . \tag{1}$$

The first covariant derivative of u_a can be uniquely decomposed into four parts:

$$u_{a;b} = {}^{(3)}\nabla_b u_a - \dot{u}_a u_b \; , \tag{2}$$

where

$${}^{(3)}\nabla_b u_a = \sigma_{ab} + \omega_{ab} + \tfrac{1}{3}\Theta h_{ab} \; , \tag{3}$$

and ${}^{(3)}\nabla_a$ is the spatial gradient operator, orthogonal to u^a. Here $\Theta = u^a{}_{;a}$ is the volume expansion, $\sigma_{ab} = \sigma_{(ab)}$ is the shear tensor ($\sigma_{ab}u^a = \sigma^a{}_a = 0$), $\omega_{ab} = \omega_{[ab]}$ is the vorticity ($\omega_{ab}u^b = 0$) and $\dot{u}^a = u_{a;b}u^b$ is the acceleration (the dot denotes a proper time derivative). It is useful to introduce a *length scale* along the fluid flow lines by the relation

$$\frac{\dot{a}}{a} = \frac{1}{3}\Theta = H \; . \tag{4}$$

When the universe is an exact FRW spacetime, H is just the usual Hubble parameter. In general, however, the evolution equation for the expansion Θ is the Raychaudhuri equation

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) - {}^{(3)}\nabla^a \dot{u}_a - \dot{u}_a \dot{u}^a + \tfrac{1}{2}\kappa(\mu + 3p) = 0 \,, \tag{5}$$

where $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$ and $\omega^2 = \frac{1}{2}\omega_{ab}\omega^{ab}$ are the shear and vorticity magnitudes and μ and p are the energy density and pressure respectively.

2.2. Matter and radiation

Fixing u^a so that it corresponds to the Landau - Lifshitz (energy) frame [18]† and considering only small deviations from equilibrium (so that the velocities of the matter and radiation relative to this frame are small), the total energy momentum tensor for matter (m) and radiation (r) is given by

$$T_{ab} = \mu u_a u_b + p h_{ab} + \pi^{(r)}_{ab} \,, \tag{6}$$

where

$$\mu = \mu_{(m)} + \mu_{(r)} \,, \quad p = \tfrac{1}{3}\mu_{(r)} \,, \tag{7}$$

and $\pi^{(r)}_{ab}$ is the anisotropic pressure of the radiation.

2.3. Component fluid equations

Relative to this frame the conservation of energy and momentum for non - interacting matter and radiation are given by [4]

$$\dot{\mu}_{(r)} + \tfrac{4}{3}\mu_{(r)}\Theta + {}^{(3)}\nabla^a q^{(r)}_a + 2q^{(r)}_a \dot{u}^a + \pi^{(r)}_{ab}\sigma^{ab} = 0 \,, \tag{8}$$

$$\dot{\mu}_{(m)} + \mu_{(m)}\Theta + {}^{(3)}\nabla^a q^{(m)}_a = 0 \,, \tag{9}$$

and

$$h^c{}_a \dot{q}^{(r)}_c + \tfrac{4}{3}\mu_{(r)}\dot{u}_a + \tfrac{1}{3}{}^{(3)}\nabla_a \mu_{(r)} + {}^{(3)}\nabla^b \pi^{(r)}_{ab} \tag{10}$$

$$+ \dot{u}^b \pi^{(r)}_{ab} + \left(\sigma_{ab} + w_{ab} + \tfrac{4}{3}\Theta h_{ab}\right) q^b_{(r)} = 0 \,, \tag{11}$$

$$h^c{}_a \dot{q}^{(m)}_c + \mu_{(m)}\dot{u}_a + \left(\sigma_{ab} + \omega_{ab} + \tfrac{4}{3}\Theta h_{ab}\right) q^b_{(m)} = 0 \,, \tag{12}$$

where

$$q^{(r)}_a = \tfrac{4}{3}\mu_{(r)} V^{(r)}_a \,, \quad q^{(m)}_a = \mu_{(m)} V^{(m)}_a \,, \quad q^{(r)}_a + q^{(m)}_a = 0 \,, \tag{13}$$

and $V^{(r)}_a$ and $V^{(m)}_a$ are the velocities of the matter and radiation relative to u^a:

$$V^{(r)}_a = u^{(r)}_a - u_a \,, \qquad V^{(m)}_a = u^{(m)}_a - u_a \,. \tag{14}$$

† In this frame the total energy flux q_a vanishes.

2.4. Total fluid equations

Because we have chosen to work in the *energy frame*, the conservation equations for the total fluid are considerably simpler than those for the individual components:

$$\dot{\mu} + h\Theta + \pi^{(r)}_{ab}\sigma^{ab} = 0 \ , \tag{15}$$

$$h\dot{u}_a + {}^{(3)}\nabla_a p + {}^{(3)}\nabla^b \pi^{(r)}_{ab} + \dot{u}^b \pi^{(r)}_{ab} = 0 \ , \tag{16}$$

where

$$h = \mu_{(m)} + \tfrac{4}{3}\mu_{(r)} \ , \tag{17}$$

is the sum of the total energy density and pressure.

2.5. FRW models

In the case of a FRW universe, $u^a_{(m)} = u^a_{(r)} = u^a$ and $\pi^{(r)}_{ab} = 0$, so the energy momentum tensor (6) necessarily reduces to the perfect fluid form

$$T_{ab} = \mu u_a u_b + p h_{ab}, \tag{18}$$

and

$$\dot{u}_a = {}^{(3)}\nabla_a p = 0 \ , \tag{19}$$

so dynamics of matter - radiation models are completely determined by the energy conservation equations

$$\dot{\mu}_{(r)} + 4\mu_{(r)}\Theta = 0 \ , \tag{20}$$

$$\dot{\mu}_{(m)} + 3\mu_{(m)}\Theta = 0 \ , \tag{21}$$

$$\dot{\mu} + h\Theta = 0 \ , \tag{22}$$

together with the Friedmann equations

$$H^2 + \frac{K}{a^2} = \tfrac{1}{3}\kappa\mu \ , \tag{23}$$

where K is the spatial curvature constant, and

$$3\dot{H} + 3H^2 + \tfrac{1}{2}\kappa\left(\mu + p\right) = 0 \ , \tag{24}$$

which is (5) specialized to a FRW model.

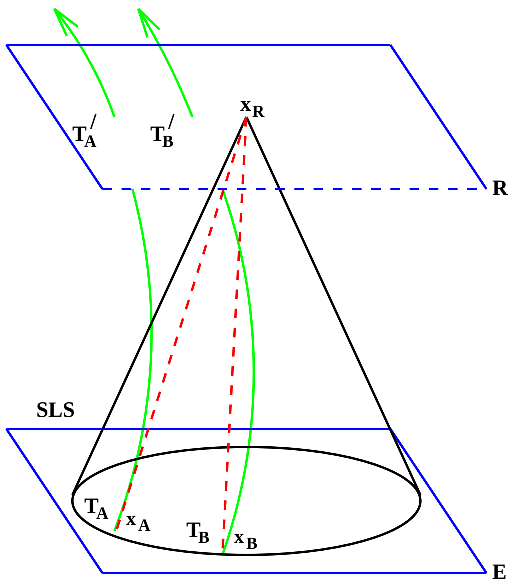

Figure 1. A diagram illustrating the geometry of the Sachs-Wolfe effect. It clearly shows that the observed temperature anisotropy at $\mathbf{x}_R$ is given by the difference in temperature of the CMB observed in different directions on the plane of the sky, corresponding to different points on the surface of last scattering (SLS).

3. A gauge-invariant measure of CMB temperature anisotropies

The aim of this paper is to derive in the simplest possible way a covariant and gauge-invariant formula that describes the observed CMB temperature anisotropy $\Delta T_R/\bar{T}_R$. This improves on earlier work by Russ *et al* [24], where a similar formula was obtained by taking first order variations of the redshift.

Before sketching this derivation, let us first clarify the difference between the gauge-invariant temperature perturbation $\delta T_R/\bar{T}_R$ and the observed temperature anisotropy, since this continues to be a source of considerable confusion in the literature.

If the point of observation is defined by the spacetime point $\mathbf{x}_R$, with coordinates x^a_R, then the temperature measured by an observer at $\mathbf{x}_R$ in a direction $\mathbf{e}_A$, with components e^a_A, can be decomposed into the average bolometric temperature $\bar{T}_R(x^a_R)$ and the covariant and gauge-invariant temperature $\delta T_R(x^a, e^a_A)$ [19]:

$$T_R(\mathbf{x}_R, \mathbf{e}_A) = \bar{T}_R(\mathbf{x}_R) + \delta T_R(\mathbf{x}_R, \mathbf{e}_A) \ , \tag{25}$$

$$\bar{T}_R(x^a_R) \qquad = \frac{1}{4\pi}\int_{4\pi} T_R(x^a_R, e^a_A)d\Omega \ . \tag{26}$$

The observed temperature anisotropy is given by

$$\frac{\Delta T_R}{\bar{T}_R} = \frac{T_R(\mathbf{x}_R, \mathbf{e}_A) - T_R(\mathbf{x}_R, \mathbf{e}_B)}{\bar{T}_R(\mathbf{x}_R)} \ , \tag{27}$$

where $\mathbf{e_A}$ and $\mathbf{e}_B$ correspond to two different directions on the plane of the sky

corresponding to two different points of emission $\mathbf{x}_A$ and $\mathbf{x}_B$ on the last scattering surface (see figure 1 above). If $\delta T_R \ll 1$ this becomes

$$\frac{\Delta T_R}{\bar{T}_R} = \frac{\delta T_R}{\bar{T}_R}(\mathbf{x}_R, \mathbf{e}_A) - \frac{\delta T_R}{\bar{T}_R}(\mathbf{x}_R, \mathbf{e}_B) \ , \tag{28}$$

since the average bolometric temperature contributions to T cancel out when subtracted at the point of observation $\mathbf{x}_R$.

4. A gauge - invariant formula for $\delta T_R/\bar{T}_R$

Having clarified the meaning of temperature perturbations and the observed temperature anisotropy, we will now proceed to derive from first principles an expression for $\delta T_R/\bar{T}_R$ in terms of the covariant quantities defined in section 2.

In principle what one should do is integrate the Louville equation in curved spacetime for a generalized gauge - invariant distribution function [28, 19] first through the decoupling phase and then from decoupling until today to obtain the perturbed spectrum of photon energies observed in a given direction $\mathbf{e}_A$ on the sky. The temperature anisotropy is then obtained by subtracting these results for two independent directions.

It turns out however, that because photons are essentially collisionless after last scattering, the complete physical content of the Louville equation is contained in the geodesic equation, so it is much simpler instead to integrate the photon energies E up null geodesics connecting points of emission on the last scattering surface with the point of observation $\mathbf{x}_R$ here and now.

Photons moves on null geodesics $x^a(\lambda)$ where λ is an affine parameter. The null vector to these geodesics is

$$p^a = \frac{dx^a}{d\lambda} \ , \qquad p^a p_a = 0 \ , \tag{29}$$

and satisfies the geodesic equation

$$p^a{}_{;b} p^b = 0 \ . \tag{30}$$

The $3+1$ decomposition of this null vector

$$p^a = E(u^a + e^a) \ , \quad e^a u_a = 0 \ , \quad e^a e_a = 1 \ , \tag{31}$$

defines the energy

$$E = -p^a u_a \tag{32}$$

of a photon relative to the four - velocity of the observer u^a and its relative (spatial) direction of motion e^a.

The temperature $T_R = T_R(\mathbf{x}_R, \mathbf{e}_E)$ of the CMB observed at the reception point R with spacetime coordinates x^a_R in a given direction $\mathbf{e}_E$ is given by

$$\frac{T_R}{T_A} = \frac{1}{1+z} = \frac{(p^a u_a)_R}{(p^a u_a)_A} \; , \tag{33}$$

where $T_A = T_A(\mathbf{x}_A)$ is the temperature at the emission point $\mathbf{x}_A$ and $z = z(\mathbf{e}_A)$ is the redshift between emission and reception.

The equation describing the variation of a photon's energy along a null geodesic (parameterized by λ) is obtained by differentiating equation (32) with respect to λ and using the geodesic equation (30). This yields

$$\frac{dE}{d\lambda} = -u_{a;b}p^a p^b \; . \tag{34}$$

Substituting for $u_{a;b}$ from equations (2) and (3) gives

$$\frac{dE}{d\lambda} = -\tfrac{1}{3}\Theta E^2 - \sigma_{ab}p^a p^b - E\dot{u}_a p^a \; . \tag{35}$$

At this point one could already integrate for E, however for the problem that we wish to discuss, it is more convenient to substitute for the expansion Θ in terms of known gauge-invariant perturbation variables. This is best done by projecting the spatial gradient of the radiation energy density $X^{(r)}_a = {}^{(3)}\nabla_a \mu_{(r)}$ along the null vector p^a:

$$p^a X^{(r)}_a = p^a \frac{\partial \mu_{(r)}}{\partial x^a} + \dot{\mu}_{(r)} p^a u_a = \frac{d\mu_{(r)}}{d\lambda} - E\dot{\mu}_{(r)} \; . \tag{36}$$

Substituting for $\dot{\mu}_{(r)}$ from the energy conservation equation for the radiation (2.3) one obtains

$$p^a X^{(r)}_a = \frac{d\mu_{(r)}}{d\lambda} + \tfrac{4}{3}\mu_{(r)} E\Theta + E\pi^{(r)}_{ab}\sigma^{ab} + E{}^{(3)}\nabla^a q^{(r)}_a + 2E\dot{u}^a q^{(r)}_a \; . \tag{37}$$

It follows that

$$E\Theta = \frac{3}{4\mu_{(r)}}\left[p^a X^{(r)}_a - \frac{d\mu_{(r)}}{d\lambda} - E\pi^{(r)}_{ab}\sigma^{ab} - E{}^{(3)}\nabla^a q^{(r)}_a - 2E\dot{u}^a q^{(r)}_a\right] \; , \tag{38}$$

and substituting this into equation (35) gives a completely general equation for the variation of a photons' energy along a null geodesic:

$$\frac{1}{E}\frac{dE}{d\lambda} - \frac{1}{4\mu_{(r)}}\frac{d\mu_{(r)}}{d\lambda} = -F \; , \tag{39}$$

where F is given by

$$\begin{aligned} F = {} & \frac{1}{4\mu_{(r)}} X^{(r)}_a p^a - \frac{E}{4\mu_{(r)}}\left[\pi^{(r)}_{ab}\sigma^{ab} + {}^{(3)}\nabla^a q^{(r)}_a + 2\dot{u}^a q^{(r)}_a\right] \\ & + \frac{1}{E}\sigma_{ab}p^a p^b + \dot{u}_a p^a \; . \end{aligned} \tag{40}$$

Integrating up a null geodesic from the point of emission $\mathbf{x}_A$ at last scattering to the point of reception $\mathbf{x}_R$ and using equation (33) one obtains an exact formula for the temperature at reception T_R:

$$\ln\left(\frac{T_R}{T_A}\right) = \frac{1}{4}\ln\left(\frac{\mu_{(r)R}}{\mu_{(r)A}}\right) - \int_A^R F d\lambda \; . \tag{41}$$

Substituting for T_R from equation (25) and using the Stephan - Boltzmann law $\mu_{(r)} = aT^4$ we obtain

$$\ln\left[1 + \frac{\delta T_R}{\bar{T}_R}\right] = -\int_A^R F d\lambda \; . \tag{42}$$

It is important to realize that apart from assuming that photons are collision free, no approximations have been made in this section so far, and the above result is therefore valid for any choice of background geometry and matter description†.

4.1. Linearization about FRW models

To obtain an expression for $\delta T_R/\bar{T}_R$ valid in an *almost* FRW model (a spacetime where these variables are small) we approach this universe from the equations valid in a general spacetime rather that adopting the standard procedure of perturbing an exact FRW model. The linearization procedure we apply consists of dropping terms such as $\pi^{(r)}_{ab}\sigma^{ab}$ in equation (40), i.e. terms which are second order in the gauge - invariant variables, retaining only those terms which are linear, for example $X^{(r)}_a$ [11]. Linearizing equation (42) following this procedure yields the following result

$$\frac{\delta T_R}{\bar{T}_R} = -\int_A^R \left(\frac{1}{4a}\mathcal{D}^{(r)}_a p^a - \frac{1}{3a}\mu_{(r)}{}^{(3)}\nabla^a V^{(r)}_a + a\sigma_{ab}p^a p^b + \dot{u}_a p^a\right) d\lambda \tag{43}$$

where $\mathcal{D}^{(r)}_a \equiv aX^{(r)}_a/\mu_{(r)}$ and we have used the normalized background (FRW) expression for the photon energy $E = 1/a$.

This formula expresses the generation of CMB anisotropies by cosmological perturbations in it's clearest form with each term having a direct physical interpretation. First it should be noted that $\mathcal{D}^{(r)}_a$, $V^{(r)}_a$ and $\dot{u}_a$ contain both a scalar and vector part, while the shear σ_{ab} is made up of contributions due to scalar, vector and tensor perturbations. Focusing on scalar perturbations, $\mathcal{D}^{(r)}_a$ and $V^{(r)}_a$ characterize density and velocity perturbations in the radiation relative to u^a, the acceleration term $\dot{u}^a$ represents possible pressure suppression effects [27] and the shear relates to perturbations in the gravitational potential.

† In the case of a FRW model, $F = 0$ and $\mu_{(r)} \propto a^{-4}$ so the standard result of $T_R/T_A = a_A/a_R$ is recovered.

We can express this result in terms of total matter variables by using the following results [4]:

$$\frac{1}{4}\mathcal{D}_a^{(r)} + a\dot{u}_a = \frac{1}{3h}\left(1 - 3c_s^2\right)\mu\mathcal{D}_a + \frac{1}{3}\frac{\mu_{(m)}^2}{h^2}S_a^{(rm)} \;, \tag{44}$$

$$V_a^{(r)} = \frac{\mu_{(m)}}{h}V_a^{(rm)} \;, \tag{45}$$

where

$$c_s^2 = \frac{4\mu_{(r)}}{3(4\mu_{(r)} + 3\mu_{(m)})} \tag{46}$$

is the speed of sound in the total fluid,

$$\mu\mathcal{D}_a = \mu_{(m)}\mathcal{D}_a^{(m)} + \mu_{(r)}\mathcal{D}_a^{(r)} \tag{47}$$

is the total perturbation in the energy density and

$$S_a^{(rm)} = \frac{1}{4}\mathcal{D}_a^{(r)} - \frac{1}{3}\mathcal{D}_a^{(m)} \;, \quad V_a^{(rm)} = u_a^{(r)} - u_a^{(m)} \tag{48}$$

are the entropy and relative velocity perturbations respectively [4].

At the time of decoupling, if the present value of the density parameter $\Omega_0 > 0.1$, the universe is matter dominated to a good approximation, so $h \to \mu_{(m)}$ and $c_s^2 \to 0$. It follows that above results reduce to

$$\frac{1}{4}\mathcal{D}_a^{(r)} + a\dot{u}_a = \frac{1}{3}\mathcal{D}_a + \frac{1}{3}S_a^{(rm)} \;, \quad V_a^{(r)} = V_a^{(rm)} \tag{49}$$

and the expression for $\delta T_R/\bar{T}_R$ becomes

$$\frac{\delta T_R}{\bar{T}_R} = \mathcal{A} - \int_A^R \left(\frac{1}{3a}\mathcal{D}_a p^a + a\sigma_{ab}p^a p^b\right) d\lambda \;, \tag{50}$$

where

$$\mathcal{A} = -\int_A^R \frac{1}{3a}\left(S_a^{(rm)}p^a - {}^{(3)}\nabla^a V_a^{(rm)}\right) d\lambda \;. \tag{51}$$

We thus have two contributions: one due to perturbations in the total energy density and pressure and the other, $\mathcal{A}$, due to the difference in the dynamical behavior of the matter and radiation density and velocity perturbations.

5. The temperature anisotropy due to gravitational potential perturbations

In this section we will deal with the contribution to CMB anisotropies due to gravitational potential fluctuations. In order to do this it is first convenient to write the

formula (50) in terms of the electric E_{ab} and magnetic H_{ab} parts of the Weyl tensor. This is done by using the two linearized Bianchi identities which relate to E_{ab} [2]:

$$\dot{E}_{ab} + 3HE_{ab} + h^f{}_{(a}\eta_{b)cde}u^c H_f{}^{d;e} + \tfrac{1}{2}h\sigma_{ab} = 0 \; , \tag{52}$$

$$a^{(3)}\nabla^a E_{ab} = \tfrac{1}{3}\kappa\mu\mathcal{D}_a \; , \tag{53}$$

to substitute for the shear and $\mathcal{D}_a$ in (50). This leads straightforwardly to the following result:

$$\begin{aligned}\frac{\delta T_R}{\bar{T}_R} = \mathcal{A} - \int_A^R \frac{1}{\mu_{(m)}} &\left[{}^{(3)}\nabla^a E_{ab}p^b - 2a\left(\dot{E}_{ab} + 3HE_{ab}\right)p^a p^b \right. \\ &+ \left. h^f{}_{(a}\eta_{b)cde}u^c H_f{}^{d;e}p^a p^b\right] d\lambda \; .\end{aligned} \tag{54}$$

This expression is closely related the formulae derived by Magueijo [20] and Durrer [5, 6].

In the case of scalar gravitational potential fluctuations the magnetic part of the Weyl tensor H_{ab} vanishes (since it only contributes to vector and tensor perturbations [2, 13]), so (54) reduces to

$$\frac{\delta T_R}{\bar{T}_R} = \mathcal{A} - \int_A^R \frac{1}{\mu_{(m)}} \left[{}^{(3)}\nabla^a E_{ab}p^b - 2a\left(\dot{E}_{ab} + 3HE_{ab}\right)p^a p^b\right] d\lambda \; . \tag{55}$$

For scalar perturbations, the electric part of the Weyl tensor (with wave number k) is related to the harmonic component of the perturbed gravitational potential $\Phi_k = \Phi_k(t)$ as follows [2]:

$$E_{ab} = \frac{k^2}{a^2}\Phi_k Q_{ab} \; , \tag{56}$$

where Q_{ab} is a covariantly defined scalar harmonic and k is the eigenvalue associated with Q_{ab} (it is the wave number if the background FRW spacetime is flat). A discussion of these harmonics and their properties is given in appendix A and [2].

Substituting (56) into (55) and using equations (A3) and (A5) in appendix A, we obtain

$$\begin{aligned}\frac{\delta T_R}{\bar{T}_R} = \mathcal{A} - 2\int_A^R \frac{1}{a^3\mu_{(m)}} &\left[\frac{1}{3}\left(3K - k^2\right)(a\Phi_k Q)' \right. \\ &- \left. K\left(a\Phi_k\right)Q' - a^3\left(a\Phi_k\right)'\left(aQ'\right)'\right] d\lambda \; ,\end{aligned} \tag{57}$$

where the prime denotes differentiation with respect to λ. In the background FRW model the energy conservation equation (21) can be integrated to give

$$\mu_{(m)} = \alpha a^{-3} \; , \quad \alpha = \mu_E a_E^3 \; , \tag{58}$$

where μ_E and a_E are the background values for the energy density and scale factor at the time of emission.

Substituting (58) into (57) and integrating the first term by parts gives

$$\frac{\delta T_R}{\bar{T}_R} = \mathcal{A} + \tfrac{2}{3}\left(3K - k^2\right)\left(\Phi_k Q\right)_E + \frac{2}{\alpha}\int_A^R \left[KQ + a^3\left(aQ'\right)'\right]\left(H\Phi_k + \dot{\Phi}_k\right) d\lambda \;, \tag{59}$$

where we have dropped the term evaluated at reception since it has no angular dependence. This result is true for a general FRW background. The $\dot{\Phi}_k$ represents the integrated or Rees - Sciama effect which is important if the potential is non - stationary, for example in open universe models.

5.1. *Large scale temperature anisotropies*

To further simplify the above problem, we will now make the standard assumptions of assuming that the background is a flat ($K = 0$) FRW model, and consider temperature anisotropies arising as a result of density perturbations on scales much larger than the Hubble radius at decoupling. In this case the Friedmann and energy conservation equations (21 - 24) lead to the following background evolution for the scale factor:

$$a = (\beta t)^{\frac{2}{3}} \;\;, \quad \beta^2 = \tfrac{3}{4}\alpha \;, \tag{60}$$

and in the matter dominated regime the potential fluctuations Φ_k satisfy the following differential equation [12]:

$$\ddot{\Phi}_k + 4H\dot{\Phi}_k = 0 \;, \tag{61}$$

which follows from (52) and the shear propagation equation when $H_{ab} = 0$. Substituting for the scale factor (60) in (59) and integrating by parts, using (61) to substitute for second derivatives in Φ_N gives the following result:

$$\frac{\delta T_R}{\bar{T}_R} = \mathcal{A} + \left(\Phi_k Q\right)_A + \tfrac{2}{3}\mathcal{H}_A\left(a\Phi_k Q_a p^a\right)_A - \tfrac{2}{9}\mathcal{H}_A^2\left(\Phi_k Q\right)_A + \tfrac{2}{3}\mathcal{H}_A\left(\tfrac{\dot{a}}{a}\right)^{-1}\left(a\dot{\Phi}_k Q_a p^a\right)_A + \textstyle\int_A^R \dot{\Phi}_k Q dt \;. \tag{62}$$

where

$$\mathcal{H}_A = \frac{k}{a_A H_A} = \left(\frac{\lambda_H}{\lambda}\right)_A \tag{63}$$

is the ratio of the Hubble scale $\lambda_H = 1/H$ to the comoving scale λ at the time of decoupling and the last term represents the integrated Sachs - Wolfe effect.

If the gravitational potential Φ_k is approximately constant and we consider scales much larger than the Hubble radius at decoupling, so that $\mathcal{H}_A \ll 1$, we recover the well known result of Sachs and Wolfe, together with additional contributions due to velocity and entropy perturbations

$$\frac{\delta T_R}{\bar{T}_R} = \mathcal{A} + \left(\Phi_k Q\right)_A \;. \tag{64}$$

Finally if we subtract this result for two independent directions (A and B), the observed temperature anisotropy $\Delta T_R/\bar{T}_R$ is obtained:

$$\frac{\Delta T_R}{\bar{T}_R} = \Delta\mathcal{A} + \Delta\left(\Phi_k Q\right)\ , \tag{65}$$

where

$$\Delta\left(\Phi_k Q\right) = \left(\Phi_k Q\right)_A - \left(\Phi_k Q\right)_B\ . \tag{66}$$

is the difference in the gravitational potential between separate points A and B on the SLS.

6. The adiabatic assumption

One of the most common assumptions made when discussing large scale CMB anisotropies is that the perturbations in the total energy density are adiabatic at the time of decoupling. Let us now consider whether or not this assumption is consistent and how it affects the results presented above. In order to achieve clarity on this issue, we need to consider (i) the large scale evolution of density, entropy and relative velocity perturbations during the collision dominated period prior to decoupling as these provide the initial conditions at the time of decoupling; (ii) how these initial conditions relate to the correct placing of the surface of last scattering and (iii) the evolution of these perturbations after decoupling.

6.1. Before decoupling

On large scales, in the matter dominated limit, the dynamics of density $\Delta_{(r)} \equiv a^{(3)}\nabla^a\mathcal{D}_a^{(r)}$, entropy $S_{(rm)} \equiv a^{(3)}\nabla^a S_a^{(rm)}$ and relative velocity $V_{(rm)} \equiv a^{(3)}\nabla^a V_a^{(rm)}$ perturbations in a photon - baryon universe are described by the following set of equations [4]:

$$\ddot{\Delta}_{(r)} + 2H\dot{\Delta}_{(r)} - \frac{1}{2}h\Delta_{(r)} = -\frac{4}{3}\left[\frac{1}{2}hS_{(rm)} - H\left(1 - \frac{h}{\mu_{(m)}}R_c\right)\dot{S}_{(rm)}\right]\ , \tag{67}$$

$$\ddot{S}_{(rm)} + H\left(\frac{4}{3}\frac{\mu_{(r)}}{h} + \frac{h}{\mu_{(m)}}R_c + 1\right)\dot{S}_{(rm)} = 0\ , \tag{68}$$

and

$$\dot{V}_{(rm)} + H\left(\frac{4}{3}\frac{\mu_{(r)}}{h} + \frac{h}{\mu_{(m)}}R_c\right)V_{(rm)} = -\frac{1}{4}\frac{1}{ha^4}\Delta_{(r)}\ , \tag{69}$$

where $R_c(t) = 1/H\tau_c$ is the ratio of the horizon size to the mean free path for photons colliding with electrons and τ_c is the mean collision time of photons with electrons.

Equations (67 - 69) can be integrated to give solutions for density, entropy and velocity perturbations in the tightly coupled regime before decoupling:

$$\Delta_{(r)} = t^{2/3} A_a \ , \tag{70}$$

$$S_{(rm)} = S_0 + B \int_0^{t_{dec}} \frac{1}{a} e^{-P(t)} dt \ , \tag{71}$$

$$V_{(rm)} = C(t) e^{-P(t)} \ . \tag{72}$$

Before decoupling $R_c(t)$ is much greater than unity since the mean collision time between photons and baryons tends to zero, and therefore $P(t) \gg 1$ in this limit. This means that the solution for $S_{(rm)}$ (71) settles down to a constant value immediately after it is provoked, and so *entropy perturbations have essentially one mode which is constant in time*. This is due to the fact that the matter and radiation are so tightly coupled that the matter cannot move relative to the radiation. This behavior can be seen by looking at the solution for $V_{(rm)}$ (72) which is exponentially driven to zero. These solutions imply that if the perturbations are initially adiabatic, as suggested by many inflationary scenarios, they will remain so until the time of decoupling.

6.2. Adiabatic perturbations at decoupling

Given that the pre - decoupling perturbation dynamics can lead to adiabatic initial conditions, let us consider whether they are compatible with the proper placing of the SLS [22, 29, 26, 10].

The correct way of placing this surface is by determining where the optical depth due to Thompson scattering is unity. This occurs, to first order, where the radiation temperature, which is equal to the matter temperature in the strongly coupled region prior to decoupling: $T_{(r)} = T_{(m)} = T$, reaches the matter ionization temperature, so the last scattering event A on each null geodesic is characterized by

$$T_A = T_{ion} \ . \tag{73}$$

Thus if we take decoupling as happening essentially instantaneous, the SLS is, to good approximation, a surface of constant radiation temperature and so by the Stefan-Boltzmann law $\mu_{(r)} = aT^4$, also one of constant radiation density:

$$\Delta T = 0 \quad \Rightarrow \quad \tilde{\mathcal{D}}_a^{(r)} = 0 \ , \tag{74}$$

where $\Delta T = T_A - T_B$ is the difference in temperature between separate points of emission x_A and x_B on the SLS (see figure 1) and $\tilde{\mathcal{D}}_a^{(r)}$ is the spatial variation of $\mu_{(r)}$ orthogonal to

the normals n^a to the surfaces of constant radiation density ($\tilde{\mathcal{D}}^{(r)}_a n^a = 0$). Transforming to the energy frame u^a, we can relate $\tilde{\mathcal{D}}^{(r)}_a$ to $\mathcal{D}^{(r)}_a$:

$$\tilde{\mathcal{D}}^{(r)}_a = \mathcal{D}^{(r)}_a + 4aHV_a \ , \quad V^a = u^a - n^a \ , \tag{75}$$

and taking its spatial divergence we obtain the corresponding result for scalar perturbations:

$$\tilde{\Delta}_{(r)} = \Delta_{(r)} + 4aHV \ , \tag{76}$$

where

$$\tilde{\Delta}_{(r)} \equiv a^{(3)}\nabla^a \mathcal{D}^{(r)}_a \ , \quad V \equiv a^{(3)}\nabla^a V_a \ . \tag{77}$$

If the perturbations are adiabatic at decoupling:

$$S_{(rm)} = 0 \Rightarrow \Delta_{(r)} = \frac{4}{3}\Delta_{(m)} \ , \quad V_{(rm)} = 0 \ , \tag{78}$$

so equation (76) becomes

$$\tilde{\Delta}_{(r)} = \frac{4}{3}\Delta_{(m)} + 4aHV \ . \tag{79}$$

Hence, for a surface of constant radiation density $\tilde{\Delta}_{(r)} = 0$, which defines the SLS, we find that:

$$\Delta_{(m)} = -3HaV \ . \tag{80}$$

Using equations (53) and we can relate $\Delta_{(m)}$ to the electric part of the Weyl tensor:

$$a^{2(3)}\nabla^{a(3)}\nabla^b E_{ab} = \tfrac{1}{3}\kappa\mu\Delta_{(m)} \ , \tag{81}$$

and combining this with (80) we obtain:

$$^{(3)}\nabla^{a(3)}\nabla^b E_{ab} = -\tfrac{\mu H}{a} V \ . \tag{82}$$

Using (56) and the results in Appendix A, the LHS of (82) can be written in terms of the harmonic components Φ_k of the perturbed gravitational potential, while the RHS can be decomposed into its harmonic components V_k (see equation 102 in [4]):

$$^{(3)}\nabla^{a(3)}\nabla^b E_{ab} = \tfrac{2}{3}\tfrac{k^4}{a^4}\left(\Phi_k Q\right) \ , \quad V = -kV_k Q \ . \tag{83}$$

Substituting these results into (82) and using equation (63) gives the following result:

$$\Phi_k Q = \tfrac{3}{2}\mathcal{H}^{-3} V_k Q \ . \tag{84}$$

It therefore follows that for adiabatic perturbations the large-scale temperature anisotropy

$$\frac{\Delta T_R}{\bar{T}_R} = \Delta\left(\Phi_k Q\right) = \tfrac{3}{2}\mathcal{H}_A^{-3}\Delta\left(V_k Q\right) \tag{85}$$

is simply related to the motion of the matter relative to the surfaces of constant temperature:

6.3. *Adiabatic perturbations after decoupling*

After decoupling, in the free propagating domain, $R_c(t) \ll 1$, so the large-scale perturbation equations (67-69) reduce to

$$\ddot{\Delta}_{(r)} + 2H\dot{\Delta}_{(r)} - \frac{1}{2}h\Delta_{(r)} = -\frac{4}{3}\left[\frac{1}{2}hS_{(rm)} - H\dot{S}_{(rm)}\right] , \tag{86}$$

$$\ddot{S}_{(rm)} + H\left(\frac{4}{3}\frac{\mu_{(r)}}{h} + 1\right)\dot{S}_{(rm)} = 0 , \tag{87}$$

$$\dot{V}_{(rm)} + \frac{4}{3}\frac{\mu_{(r)}H}{h}V_{(rm)} = -\frac{1}{4}\frac{1}{ha^4}\Delta_{(r)} , \tag{88}$$

and in the matter dominated limit they can be integrated to give the following solutions

$$\Delta_{(r)} = At^{2/3} , \tag{89}$$

$$S_{(rm)} = S_0 + Bt^{1/3} , \tag{90}$$

$$V_{(rm)} = Ct . \tag{91}$$

These solutions demonstrate that after decoupling generic density perturbations do not satisfy the adiabatic condition $S_{(rm)} = V_{(rm)} = 0$. This is due to the fact that the average velocity of the radiation does not proceed along geodesics, while the matter does. Thus any perturbation that starts off adiabatic at last scattering will not remain so.

7. Conclusion

In this paper we have calculated a fully covariant formula for the CMB temperature anisotropy improving on earlier work by Russ *et al* [24]. This formulation has a number of distinct advantages over the more standard approaches as it is independent of gauge conditions, non-local splittings of spacetime, and related Fourier decompositions of perturbations around a FRW metric. Furthermore the results are relatively simple and easy to interpret. For scalar perturbations we recovered the dominant Sachs-Wolfe term, together with the Rees-Sciama effect which contributes to large scale CMB anisotropy only if the perturbations to the gravitational potential are non-stationary.

We also examined the validity of the assumption that the density perturbations are adiabatic at decoupling and showed that if the surface of last scattering is correctly placed and the background is assumed to be a flat ($K = 0$) FRW model, then the

scalar (Sachs - Wolfe) contribution to large scale CMB anisotropies may be interpreted as being due to the motion of matter relative to the surfaces of constant temperature which define the surface of last scattering on scales where the instantaneous decoupling approximation applies.

Acknowledgments

I thank George Ellis, Tim Gebbie, Roy Maartens, Marco Bruni, William Stoeger, Alan Coley, Dick Bond, Bernard Carr and the referees for useful discussions. This work was supported by the FRD (South Africa) and a Dalhousie Postdoctoral Fellowship.

Appendix A. Covariantly defined harmonics

In the standard approach to cosmological perturbations [1, 17] a harmonic decomposition of the perturbation variables is usually carried out using harmonics which are eigenfunctions of the Laplace - Beltrami operator on the three - hypersurfaces of constant curvature i.e. on the homogeneous spatial sections of FRW universes. In the covariant approach, the fluid four - velocity u^a is emphasized rather than an arbitrarily chosen spatial slicing, and quantities are defined by projecting orthogonal to u^a using the projection tensor h_{ab}. Covariant harmonics are therefore defined through operators constructed with the spatial (orthogonal to u^a) derivative ${}^{(3)}\nabla_a$ which are covariantly constant along the fluid flow lines (i.e. independent of proper time). In this section we will focus on scalar harmonics Q which are the eigenfunctions of the covariantly defined Laplace - Beltrami operator [14]:

$$ {}^{(3)}\nabla^2 Q = -\frac{k^2}{a^2} Q \ , \tag{A1} $$

where k is a comoving (i.e constant) eigenvalue. If ν is a non - negative real wavenumber, then for a flat background ($K = 0$), it is associated with the physical wavelengths $\lambda = 2\pi a/\nu$, since in this case $k = \nu$, however for open models ($K = -1$) the spectrum of eigenvalues is given by $k^2 = \nu^2 + 1$ [15].

The scalar harmonic Q can be used to define a vector

$$ Q_a = -\frac{a}{k} {}^{(3)}\nabla_a Q \tag{A2} $$

and a trace - free symmetric tensor

$$ Q_{ab} = \frac{a^2}{k^2} {}^{(3)}\nabla_b {}^{(3)}\nabla_a Q + \tfrac{1}{3} h_{ab} Q \ . \tag{A3} $$

These harmonics are defined in order to have

$$ \dot{Q} = \dot{Q}_a = \dot{Q}_{ab} = 0, \tag{A4} $$

so that they are covariantly constant along u^a. Finally, by taking the spatial divergence of (A3) the following relation between Q_{ab} and Q_a is obtained [2]

$$a^{(3)}\nabla^b Q_{ab} = -\tfrac{2}{3}k^{-1}\left(3K - k^2\right) Q_a \ . \tag{A5}$$

This is needed in section 5.

References

[1] J M Bardeen 1980, *Phys. Rev.* D, **22**, 1882.

[2] M Bruni, P K S Dunsby and G F R Ellis 1992, *Astrophys. J.*, **395**, 34.

[3] P K S Dunsby 1991, *Classical and Quantum Gravity* **8**, 1785.

[4] P K S Dunsby, M Bruni and G F R Ellis 1992, *Astrophys. J.*, **395**, 54.

[5] R Durrer 1994, *Fundamentals in Cosmic Physics*, **15**, 209.

[6] R Durrer and Z-H Zhou 1995, astro-ph/9508016.

[7] J Ehlers 1961, *Abh. Mainz Akad. Wiss. Lit. (Math. Nat. Kl.)* **11**, 1.

[8] G F R Ellis 1971, in *General Relativity and Cosmology*, Proceedings of XLVII Enrico Fermi Summer School, ed Sachs, R. K. (New York Academic Press).

[9] G F R Ellis 1973, in *Cargese Lectures in Physics, Volume 6*, Ed. Schatzmann, E. (Gordon and Breach).

[10] G F R Ellis and P K S Dunsby 1997, CBR Anisotropy: The Case of Adiabatic Perturbations, University of Cape Town Preprint.

[11] G F R Ellis and M Bruni 1989, *Phys. Rev.* D, **40**, 1804.

[12] G F R Ellis, J Hwang and M Bruni 1989, *Phys. Rev.* D **40**, 1819.

[13] S W Goode 1989, *Phys. Rev.* D **39**, 2992.

[14] S W Hawking 1966, *Ap. J.* **145**, 544.

[15] E R Harrison 1967, *Rev. Mod. Phys.* **39**, 862.

[16] C J Hogan, N Kaiser and M J Rees 1982, *Philos. Trans. R. Soc. London* **A307**, 97.

[17] H Kodama and M Sasaki 1984, *Prog. Theor. Phys.* **78**, 1.

[18] L Landau and E Lifshitz 1963, Fluid Mechanics (Oxford: Pergamon).

[19] R Maartens, G F R Ellis and W R Stoeger 1995, *Phys. Rev.* D **51**, 5125.

[20] J C R Magueijo 1993, *Phys. Rev* D **47**, R353.

[21] T Padmanabhan 1993, *Structure Formation in the Universe* (Cambridge University Press), pp 222-247.

[22] M Panek 1986, *Phys. Rev.* D **34**, 416.

[23] P J E Peebles 1980, *The Large-Scale Structure of the Universe* (Princeton University Press), pp. 363-372.

[24] H Russ, M Soffel, C-M Xu, and P K S Dunsby 1993, *Phys. Rev.* D **48**, 4552.

[25] R K Sachs and A M Wolfe 1967, *Astrophys. J.* **147**, 73.

[26] W R Stoeger, G F R Ellis, and B G Schmidt 1991, *Gen. Relativ. Gravit.* **23**, 1169.

[27] W R Stoeger, G F R Ellis and C-M Xu 1994, *Phys Rev.* D **49**, 1845.

[28] W R Stoeger , R Maartens and G F R Ellis 1995, *Astrophys. J.* **443**, 1.

[29] W R Stoeger, C-M Xu, G F R Ellis, and M Katz 1995, *Astrophys. J.* **445**, 17.